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LETTER TO THE EDITOR

Random walks on multifractal lattices

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Abstract. The properties of random walks on a square lattice with multifractal distributions of site-residence probabilities have been explored using computer simulations. Random walks on such substrates have well defined fractal dimensionalities D_w (obtained from the dependence of $\overline{R^2}$ on N where $\overline{R^2}$ is the mean square displacement from the origin and N is the number of steps in the walk) which are larger than two. Results obtained from these simulations indicate that more than two exponents (D_w and D_s where D_s is the fracton or spectral dimensionality) are needed to describe the properties of these walks. The algorithms used in this work provide a convenient way for generating walks with a fractal dimensionality greater than two and are being used to extend the scope of the diffusion-limited aggregation models.

Knowledge of the properties of random walks provides a basis for understanding an extremely broad range of phenomena in physics, chemistry, biology and other areas. For this reason random walks have been intensively investigated throughout this century. In recent years, considerable attention has been focused on the properties of random walks on fractals (Mandelbrot 1982) and other non-Euclidean systems. This work has been motivated by the realisation that many of the unique properties of fractals are a direct consequence of the properties of random walks on these structures, which can be described in terms of scaling exponents such as the fractal dimensionality of the random walk and its fracton or spectral dimensionality (Alexander and Orbach 1982). Here computer simulations are used to explore some of the properties of random walks on multifractal lattices which can be described in terms of an infinite family of scaling exponents and fractal dimensionalities (Mandelbrot 1974, 1982, Halsey *et al* 1986).

The multifractal lattices used in this work are illustrated in figure 1. In the first stage of construction, four numbers (which may be regarded as probabilities) P_1, P_2, P_3 and P_4 are randomly associated with the four quadrants of a square lattice (figure 1(a)). In the next stage each of the quadrants is divided into four smaller quadrants and the probabilities associated with these quadrants are multiplied by P_1, P_2, P_3 and P_4 (in random order) (figure 1(b)). Starting with a lattice containing $2^n \times 2^n$ sites this process is continued for n generations until a probability or measure ($\mu(x)$) is associated with each of the lattice sites. In general, the probability associated with a lattice site will have the form $P_1^i P_2^j P_3^k P_4^l$ with $i+j+k+l=n$. Figure 1(c) shows a typical example based on the generator $P_1=1$, $P_2=1$, $P_3=0.5$ and $P_4=0.5$ with $n=9$ generations (512×512 sites). In the limit $n\to\infty$ the procedure outlined above defines a fractal measure on the two-dimensional space which can be described in terms of a continuous spectrum of singularities of type α, each supported on a fractal subset with a fractal dimensionality of $f(\alpha)$ (Halsey *et al* 1986). Multifractals of the type illustrated in figure 1 have been used for other purposes such as the description of turbulence (Mandelbrot 1974, Frisch *et al* 1978, Shertzer and Lovejoy 1983, Benzi *et al* 1984, Lovejoy and Schertzer 1986).

For multifractals of the type used as substrates in this work, the function $f(\alpha)$ can be obtained from the asymptotic ($n \to \infty$) form of the probability histogram. The simplest case $P_1 = 1$, $P_2 = R_3$, $P_3 = R_3$, $P_4 = R_3^2$ ($R_3 < 1$) leads to a log-binomial distribution and in this case the function $f(\alpha)$ is given by

$$g(x) = -4[x \ln x + (1-x)\ln(1-x)]/\ln 2 \tag{1}$$

for $0 < x < 1$ where

$$f(A(x+B)) = g(x) \tag{2}$$

with $A = -2 \ln R_3/\ln 2$ and $B = -2\ln(1+R_3)/\ln 2$ (Meakin 1987).

To explore the properties of random walks on these lattices, lattice sites (with coordinates i and j) are selected at random with a probability proportional to the value of the probability measure ($\mu(i,j)$) associated with that lattice site. This provides an origin for the walk. The probability that the random walker will move into the nearest-neighbour site at the position (k, l) is equal to 1.0 if $\mu(k,l) \geqslant \mu(i,j)$ and is proportional to $\mu(k,l)/\mu(i,j)$ if $\mu(k,l) < \mu(i,j)$. According to these rules the random walk corresponds to the trajectory of a particle in a standard Monte Carlo simulation (Metropolis *et al* 1953) which might be carried out to estimate the thermodynamic properties of the system. However, steps during which the particle does not move are not explicitly included.

Simulations were carried out using two types of multifractal substrates (type I and type II). For type I, $P_1 = P_2 = 1$ and $P_3 = P_4 = R_1$. For type II, $P_1 = 1$, $P_2 = R_2$, $P_3 = R_2^2$ and $P_4 = R_2^3$. The simulations were carried out using two-dimensional square lattices containing 1024×1024 sites and 100 walks each containing 100 000 steps were carried out on each lattice. For each value of the parameter R_1 (for type I lattices) or R_2 the simulations were repeated approximately 600 times using different (randomly generated) lattices.

The exact enumeration method (Majid *et al* 1984) could easily be adapted to these multifractal lattices. This method was not used because a large fraction of all possible walks have very low probabilities on the multifractal lattices and because it was believed to be more important to sample a large number of different multifractal lattices and different origins on these lattices than to enumerate all of the possible walks from a given origin. However, it is possible that the exact enumeration method could considerably reduce the statistical uncertainties and this is a subject for further investigation.

Figure 2 shows the dependence of $\ln \overline{R^2}$ on $\ln N$ where $\overline{R^2}$ is the mean square distance travelled by the random walker from its origin after N steps for several values of the parameter R_1. These log-log plots are almost perfectly linear (except for very small values of R_1). From the definition of the random walk dimensionality, D_w,

$$\overline{R^2} \sim N^{2\nu} \sim N^{2/D_w} \tag{3}$$

the values for D_w given in table 1 were obtained. Figure 3 shows the dependence of the number of sites visited, N_s, on the number of steps in the random walk. Here we have plotted $\ln N_s$ against $\ln(N/\ln N)$. For a random walk on a two-dimensional lattice with uniform measure

$$N_s \sim N/\ln N \tag{4}$$

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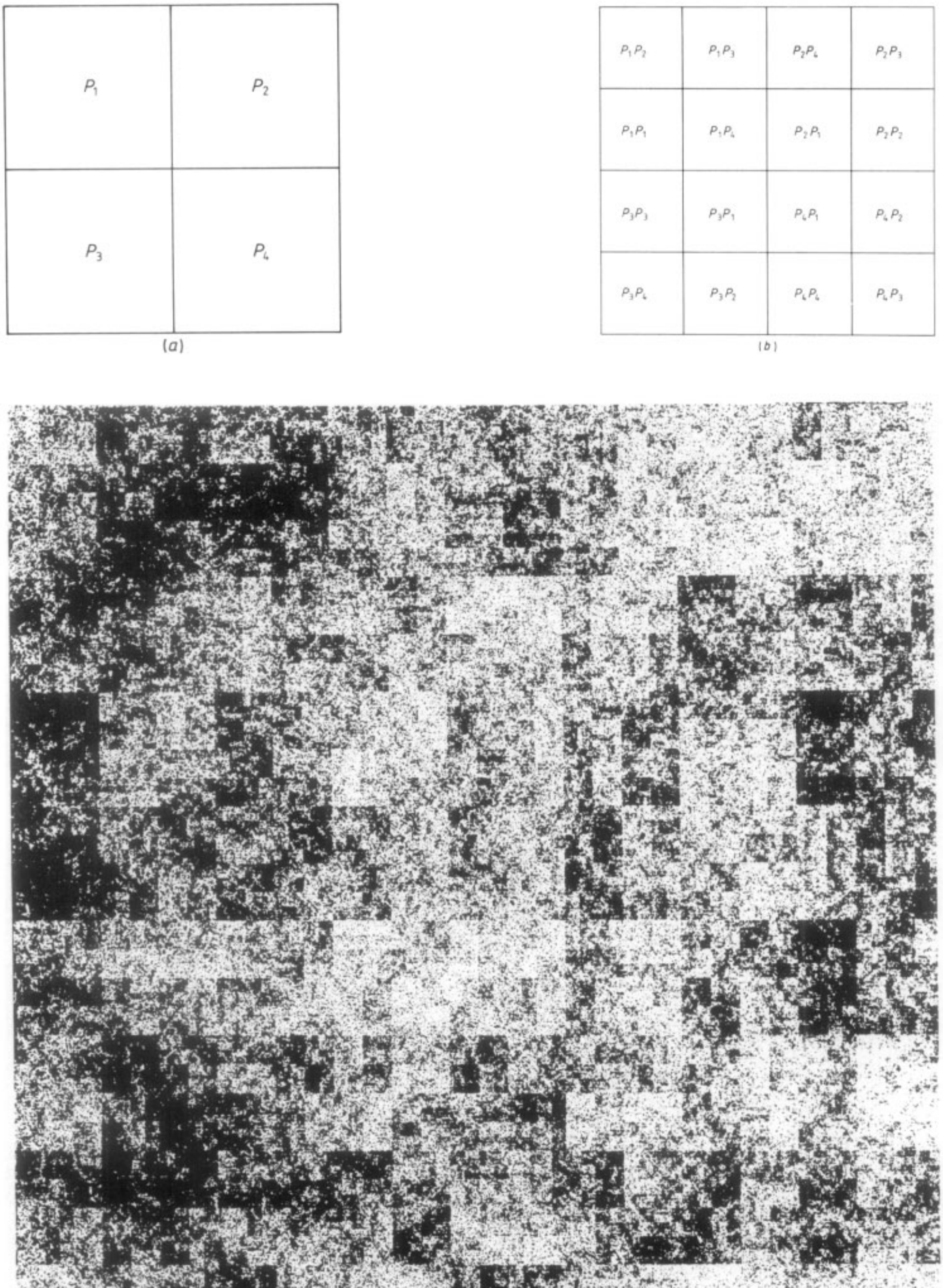

Figure 1. Generation of the multifractal substrates. (a) The generator for the fractal measure. (b) The second stage in the construction process. (c) The ninth stage for a 512×512 lattice using the generator $P_1 = P_2 = 1$, $P_3 = P_4 = R$. The density of points in each lattice site is proportional to the probability measure associated with that site.

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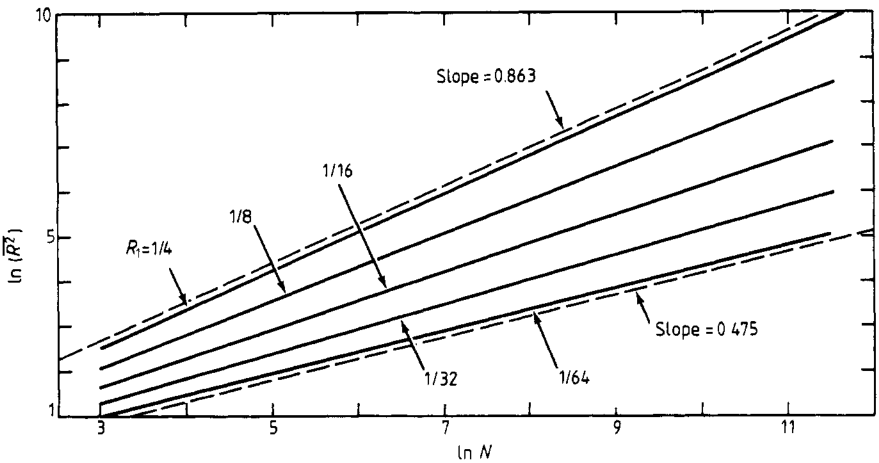

Figure 2. Dependence of the mean square displacement of the random walkers from their origins as a function of the number of steps, N, in the random walk. These results were obtained using substrates with a probability measure obtained from generators of the type $P_1 = P_2 = 1$, $P_3 = P_4 = R_1$.

Table 1. Effective exponents obtained for random walks on multifractals of type I ($P_1 = P_2 = 1$, $P_3 = P_4 = R_1$) for various values of R_1.

R_1	$2/D_w$	ξ'	ξ	ω	η
$\frac{1}{4}$	0.8641	0.886	0.796	1.245	0.773
$\frac{1}{8}$	0.7514	0.771	0.693	1.332	0.587
$\frac{1}{16}$	0.6447	0.663	0.587	1.401	0.472
$\frac{1}{32}$	0.5534	0.574	0.515	1.603	0.384
$\frac{1}{64}$	0.4926	0.514	0.462	1.727	0.295

(Weiss and Rubin 1983). For values of R_1 which are close to 1.0, plotting ln N_s against ln(N/ln N) gives a more nearly linear behaviour than plotting ln N_s against ln N. Under these conditions we find that

$$N_s \sim (N/\ln N)^{\xi'} \tag{5}$$

and that $\xi' \simeq 2/D_w$. Since the corrections to the asymptotic scaling behaviour are not known, the dependence of ln N_s on ln N has also been determined and effective values of the exponents ξ' and ξ defined by

$$N_s \sim N^{\xi} \tag{6}$$

are given in table 1 for walks of length $1000 \leqslant N \leqslant 100\,000$ steps.

It seems quite probable that equation (6) rather than equation (5) describes the asymptotic behaviour of these random walks (i.e. asymptotic logarithmic corrections occur only when the fractal dimensionality of the walk and the substrate are equal). Unfortunately, it is not possible to distinguish between equations (5) and (6) (or other forms for the corrections to scaling) on the basis of these simulations alone.

The probability F_N, that once a site has been visited by the random walker it will first return to that site after N steps, has also been measured. Figure 4 shows the

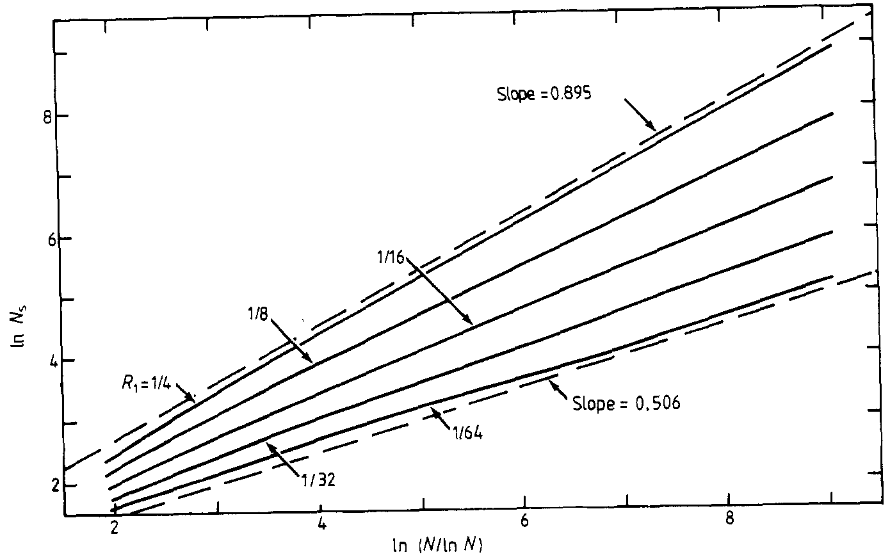

Figure 3. Dependence of the total number of sites visited, N_s, on the walk length, N, for random walks on multifractal substrates with generators of the type illustrated in figure 1 with $P_1 = P_2 = 1$ and $P_3 = P_4 = R_1$.

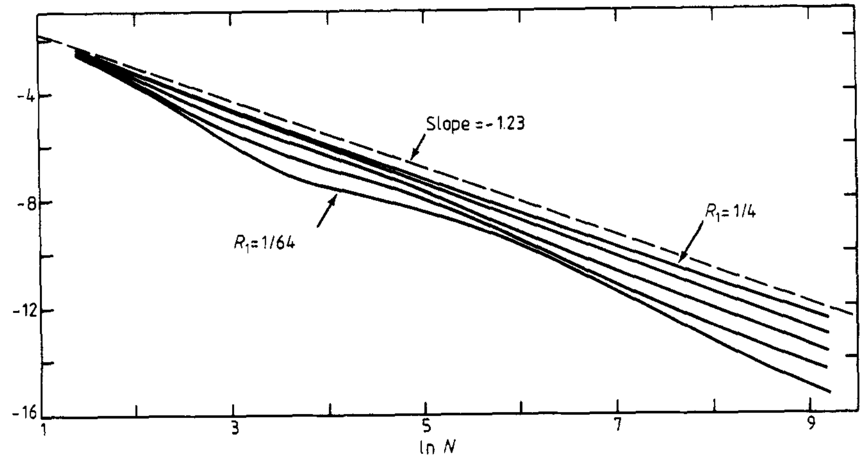

Figure 4. This figure shows the dependence of ln F_N on ln N for random walks on multifractal substrates of the type illustrated in figure 1 with $P_1 = P_2 = 1$ and $P_3 = P_4 = R_1$. F_N is the probability that a random walker will first return to a previously occupied site after N steps.

dependence of ln F_N on ln N. For large N the results are consistent with an asymptotic scaling reaction of the form $F_N \sim N^{-\omega}$ and effective values for the exponent ω (for $1000 \leq N \leq 10\,000$) are given in table 1.

The probability G_N that a previously visited site will be revisited again after N steps has also been measured. In this case every return is counted, not just the first return. The simulation results are consistent with an asymptotic power law of the form $G_N \sim N^{-\eta}$. The dependence of ln G_N or ln N is much more linear for small values of R_1 or R_2 than it is for large values (similar behaviour is shown for F_N in figure 4).

The values given for η in tables 1 and 2 were obtained by least squares fitting straight lines to the dependence of $\ln G_N$ on $\ln N$ over the range $1000 \leq N \leq 10\,000$ steps.

Because significant corrections are expected, the exponents given in tables 1 and 2 should be regarded as effective exponents for the range of length scales over which they were measured. The best power-law behaviour was found for the dependence of R^2 on N and the values given for the corresponding exponent (D_w) in tables 1 and 2 are probably the most reliable. At present, the form of the corrections to scaling for the other quantities measured in this work is not known and a much more extensive study than that reported here would be needed to determine the nature of these corrections.

Table 2. Effective exponents obtained for random walks on multifractals of type II ($P_1 = 1$, $P_2 = R_2$, $P_3 = R_2^2$, $P_4 = R_2^3$) for various values of R_2.

R_2	$2/D_w$	ξ'	ξ	ω	η
$\frac{3}{4}$	0.9694	0.985	0.885	1.189	0.916
$\frac{1}{2}$	0.8373	0.856	0.769	1.265	0.699
$\frac{1}{4}$	0.5830	0.603	0.541	1.465	0.417
$\frac{1}{8}$	0.4079	0.433	0.388	1.612	0.250
$\frac{1}{16}$	0.3008	0.332	0.297	1.733	0.168

The exponent ξ relating the number of sites visited to the number of steps in this walk is often expressed in terms of the spectral or fracton dimensionality as (Rammal and Toulouse 1983)

$$\xi = D_s/2 \qquad \text{or} \qquad D_s = 2\xi. \tag{7}$$

The observation that $\xi \simeq 2/D_w$ implies that $D_s \simeq 4/D_w$ which with the definition $D_s = 2D/D_w$ gives $D = 2$ for the effective fractal dimensionality of the substrate. The spectral dimensionality can also be measured from the distribution of first returns to a previously visited site (Alexander and Orbach 1982, Rammal and Toulouse 1983). In this case the exponent ω is related to D_s by $\omega = D_s/2$ or $D_s = 2\omega$ and this implies (with equation (5)) that $\xi = \omega$. It is clear from the results given in tables 1 and 2 that $\omega \neq \xi$ (except in the limit $R \to 1$ corresponding to an ordinary two-dimensional random walk). Consequently, it seems that more than two exponents (D_w and D_s, for example) are needed to describe these random walks. For a random walk on a fractal the exponent η which describes the probability of return to the origin after N steps should also be equal to ξ and ω.

To measure the exponent ω a record is kept of the number of steps in the walk at which each of the sites was last visited. When a previously visited site is revisited N steps later, the contribution of this N-step 'loop' is added to F_N and the information concerning the number of steps in the walk at which this site was last visited is updated. The updating is stopped 10 000 steps before the end of the walks so as not to introduce a bias against long loops (less than 10 000 steps long) which would not be finished before the walk had ended. A similar procedure was used to obtain G_N and the exponent η. Now the site is identified by the step in the walk at which it was first visited and a contribution to G_N is added each time this site is revisited N steps after it was first visited. Unfortunately, this procedure distorts the shape of G_N since sites found for the first time early in the walk will tend to be those with high probabilities and those visited for the first time towards the end of the walk will tend to have relatively low probabilities. The contribution of reference sites with low probabilities will be overemphasised. The effect of this bias is not clear, but it means that the results shown in tables 1 and 2 are not inconsistent with the idea that $\eta = \xi$.

A smaller distortion in the shapes of both F_N and G_N results from the fact that we have not taken into account the number of attempts which a random walker makes to move from a particular site before succeeding. However, it seems unlikely that this will be sufficient to change the asymptotic ($N \to \infty$) dependence of F_N or G_N on N. However, it could be responsible for the deviations from linearity observed in the dependence of ln F_N or ln N for large values of R (figure 4).

The unusual properties of the random walks investigated here result from the multifractal correlations in the substrate. These correlations are well understood; they can be quantitatively described in terms of the language of multifractals or fractal measures (Mandelbrot 1974, Halsey *et al* 1986). Because of this, and because of the fact that these walks can easily be generated and the properties continuously varied, it is likely that this model will prove to be useful in exploring the properties of correlated walks. Here we have focused attention on the most simple ensemble-averaged properties. It is likely that the distribution of properties within the ensemble will be quite different from those found in uncorrelated walks. For ordinary random walks the moments of the end-to-end distance are given by

$$\overline{R^n} \sim N^{n\nu}. \tag{8}$$

It is possible that a more general description will be needed for walks on multifractal substrates in which an infinite family of exponent ν_n will be needed, i.e.

$$\overline{R^n} \sim N^{n\nu_n}. \tag{9}$$

Walks originating in regions of very high probability may tend to be localised in that region and have a high fractal dimensionality whereas walks originating in regions of low probability may tend to follow a path of lower dimensionality as they 'seek' regions of higher probability. This possibility will be explored in the near future. At present, random walks generated using the model described above are being used to extend the scope of the DLA model of Witten and Sander (1981).

After this work had been completed, I learned that a similar one-dimensional model had recently been investigated analytically and using computer simulations by Weissman and Havlin (1987).

The ideas and motivation for this work were developed during a CECAM (Centre Européen pour le Calcul Atomique et Moléculaire) workshop on multifractals and during a visit to the Institute of Physics, University of Oslo. I would like to thank J Feder and T Jossang for their hospitality and encouragement.

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